# ASSESSMENT OF A FOUR MOMENTS AND PEARSON DISTRIBUTION APPROACH FOR RISK ANALYSIS IN CONSTRUCTION ESTIMATES AND ECONOMICS

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ABSTRACT

The use of a standard four moments approach has been advocated in the past for risk analysis of construction project estimates and economic measures. The approach calls for the computation of the four central moments of the performance function using an approximated Taylor series expansion. The four moments are then fitted to a distribution from the Pearson distribution family. In instances where this approach yields accurate results, it can offer a significant alternative to simulation. However, in instances of high non-linearity of the performance measures, results from such an approach do not match well with those from Monte-Carlo simulation. This paper investigates this issue and introduces an expanded four moment formulation that provides a more accurate estimate of the moments than that offered by the standard moment approach. A number of examples using summation, multiplication, division, and exponentiation were used to investigate and compare the three approaches, with attention being limited to independent variables (i.e. no correlation). The results of the analysis demonstrate that the expanded four moments give better estimates than the standard moment approach. However, it is shown that when performance measures involve variables of higher exponent/power, both approaches produce results significantly different from those obtained by simulation.

## KEY WORDS

risk analysis; probabilistic models; simulation; heavy civil

## INTRODUCTION

For risk analysis of construction estimates and project economics, the moment approach can be used to determine the first four moments of a performance measure/system function given knowledge of the probabilistic characteristics of the primary variables of the system function (Siddall 1972). A probability distribution is then characterized from one of the distribution families; commonly the Pearson distribution family is used. The derived moments of the system function, however, are not the exact moments since contributions from moments

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higher than the fourth level of the primary variables are generally not available and thus the moment estimating equations do not include all applicable terms. Introduced in this paper is an expanded four moment formulation for system functions. The virtue of this formulation is investigated through a number of cases which explore the closeness of the four moments/ distributions to the exact/near exact ones obtained by large scale Monte-Carlo simulation.

## STANDARD FOUR MOMENTS AND PEARSON DISTRIBUTIONS

Following Hahn and Shapiro (1967) and Siddall (1972), the first four moments of a system function can be obtained using the first four moments of the primary variables in the system function. The moment approach involves expanding the system function using a multivariate Taylor series, generally up to the second order, around the means of the primary variables. Then, by taking expectations of the expanded system function, the four moments of the system function are obtained, defined here as standard four moments. Let Z to represent a system function $f(\boldsymbol{x})$ in which $\boldsymbol{x}$ is a vector of random variables $x_i$ where $i = 1,2, \ldots, n$ (i.e. $x_1, x_2, \ldots, x_n$); $f(\bar{\boldsymbol{x}})$ is $Z$ calculated at the mean values of $\boldsymbol{x}$; $f_i'$ and $f_i''$ are the 1st and 2nd partial derivatives of $Z$ with respect to the $i$-th variable in $\boldsymbol{x}$; $\mu_1{`}_i$, $\mu_{2_i}$, $\mu_{3_i}$, and $\mu_{4_i}$ are the expected value, and 2nd to 4th central moments of the variable $i$; $\mu{`}_1(Z)$, $\mu_2(Z)$, $\mu_3(Z)$ and $\mu_4(Z)$ are the expected value, 2nd to 4th central moments of $Z$:

$$\mu_1'(Z) = f(\bar{\mathrm{x}}) + \frac{1}{2} \cdot \sum_{i=1}^{n} f_i'' \cdot \mu 2_i \tag{1}$$

$$\mu_2(Z) = \sum_{i=1}^{n} f_i'^2 \cdot \mu 2_i + \sum_{i=1}^{n} f_i' \cdot f_i'' \cdot \mu 3_i \tag{2}$$

$$\mu_3(Z) = \sum_{i=1}^{n} f_i'^3 \cdot \mu 3_i \tag{3}$$

$$\mu_4(Z) = \sum_{i=1}^{n} f_i'^4 \cdot \mu 4_i + 6 \cdot \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_i'^2 \cdot f_j'^2 \cdot \mu 2_i \cdot \mu 2_j \tag{4}$$

Since probability distributions can be characterized by four or more parameters, e.g. their location (mean), dispersion (variance), skewness and kurtosis, families of distributions were developed such that a specific distribution could be characterized from within the family. One of the common distribution families is the Pearson distribution family which encompasses a wide range of shapes and includes several common distributions such as the Normal, LogNormal, Beta, Gamma, Exponential and Uniform (Johnson et al. 1963; Elderton and Johnson 1969; Hahn and Shapiro 1967; Johnson et al. 1994; Ord 1972). Thus, given estimates of the first four moments of the system function, probabilistic statements can be made using the best fit Pearson distribution.

## THE ANALYSIS

The four moments of a system function, however, are not the exact moments since contributions from moments higher than the fourth level of the primary variables are not included and because not all the moment terms of the primary variables are included, as well. Interestingly, there is little discussion in the literature as to the accuracy of the standard moment expressions or the magnitude of the terms dropped. For example, the authors own experience in dealing with moment approximations is that neglect of the skewness term for the independent variables in the approximation for the variance of the system function can introduce significant error. Including higher moments, however, would require more computations and more information than what reasonably can be obtained for risk analysis. Under such circumstances, the current work checks the reasonableness of the four moment approach as used in risk analysis and introduces an expanded four moment formulation.

First, more accurate expressions for the first four moments were derived. They are called expanded four moments and are presented in Appendix A. The new derivation includes all the terms of the moments of the primary variables as well as the second partial derivatives of the system function. Second, in order to achieve better estimates of probabilities, rather than using the double-entry tables for Pearson distributions (Elderton and Johnson 1969), the current work follows a characterization process that gives probabilities at any required value, where given the four moments of a system function, a Pearson distribution is selected, its parameters are estimated and its closed form is used to determine probabilistic estimates. Third, a number of cases as identified in Table 1 were established for the analysis where for each case the standard four moments and the new expanded four moments and their Pearson distribution were compared to those of a large scale Monte Carlo simulation which involved 100,000 iterations. The cases includes simple formulations generally used in construction cost and time calculations, and other cases of high non-linearity and higher order of power which were included for the analysis of the approaches under worst conditions.

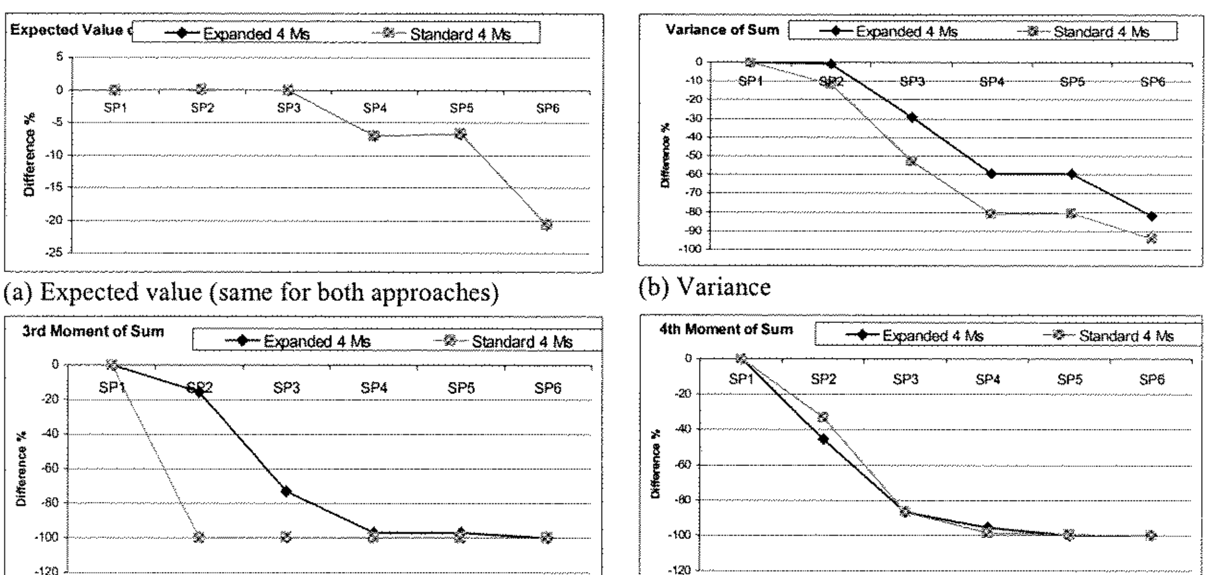
Table 1: Cases for the analysis

| **Sum** | | **Product** | | **Division** | | **Exponential** | |
|---|---|---|---|---|---|---|---|
| SP1 | $x_1^1+x_2^1+x_3^1+x_4^1+x_5^1$ | P1 | $x_1^1 \cdot x_2^1$ | D1 | $x_1^1 \cdot x_2^{-1}$ | E1 | $x_1^1 \cdot e^{x_2}$ |
| SP2 | $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2$ | P2 | $x_1^1 \cdot x_2^1 \cdot x_3^1$ | D2 | $x_1^1 \cdot x_2^{-1} \cdot x_3^{-1}$ | E2 | $x_1^1 \cdot e^{-x_2}$ |
| SP3 | $x_1^3+x_2^3+x_3^3+x_4^3+x_5^3$ | P3 | $x_1^1 \cdot x_2^1 \cdot x_3^1 \cdot x_4^1$ | D3 | $x_1^1 \cdot x_2^{-1} \cdot x_3^{-1} \cdot x_4^{-1}$ | E3 | $x_1^2 \cdot e^{x_2}$ |
| SP4 | $x_1^4+x_2^4+x_3^4+x_4^4+x_5^4$ | P4 | $x_1^2 \cdot x_2^2$ | D4 | $x_1^2 \cdot x_2^{-2}$ | E4 | $x_1^2 \cdot e^{-x_2}$ |
| SP5 | $x_1^1+x_2^2+x_3^3+x_4^4$ | P5 | $x_1^2 \cdot x_2^2 \cdot x_3^2$ | D5 | $x_1^2 \cdot x_2^{-2} \cdot x_3^{-2}$ | E5 | $x_1^2 \cdot e^{x_2} \cdot e^{x_3}$ |
| SP6 | $x_1^1+x_2^2+x_3^3+x_4^4+x_5^5$ | P6 | $x_1^2 \cdot x_2^2 \cdot x_3^2 \cdot x_4^2$ | D6 | $x_1^2 \cdot x_2^{-2} \cdot x_3^{-2} \cdot x_4^{-2}$ | E6 | $x_1^2 \cdot e^{-x_2} \cdot e^{-x_3}$ |
| | | P7 | $x_1^3 \cdot x_2^3$ | D7 | $x_1^3 \cdot x_2^{-3}$ | E7 | $x_1^2 \cdot e^{-x_2} + x_3^2 \cdot e^{-x_4}$ |
| | | P8 | $x_1^3 \cdot x_2^3 \cdot x_3^3$ | D8 | $x_1^3 \cdot x_2^{-3} \cdot x_3^{-3}$ | E8 | $x_1^3 \cdot e^{-x_2} + x_3^3 \cdot e^{-x_4}$ |
| | | P9 | $x_1^3 \cdot x_2^3 \cdot x_3^3 \cdot x_4^3$ | D9 | $x_1^3 \cdot x_2^{-3} \cdot x_3^{-3} \cdot x_4^{-3}$ | | |

### CASE #1 SUM OF VARIABLES

A system function which involves the sum of variables is reflective of construction estimates and present values computations. Table 1 shows six cases for the sum of variables where Normal N(10, 5) distribution is used for the variables. N(10,5) was chosen to explain the extreme or to how far the difference, if any, between the approaches could go since the variance and the coefficient of variation are high. Figure 1 shows for each case the difference percentage of the variance, 3rd moment and 4th moment of both the standard moment approach and the expanded four moments when compared to the Monte Carlo simulation. The results in the figure show that: (1) the increase in the exponential power of the variables increases the deviation of the moments from those of Monte Carlo simulation; (2) the 4th moment (kurtosis) is not reasonably estimated with both moment approaches except for cases of lower order of power (3) the expanded moment derivation gives better estimates of the 2nd and 3rd moments (variance, skewness); and, (4) the expected value is generally reasonably estimated by both moment approaches. It could be concluded that performance measures, such as construction time and cost would be reasonably estimated using the expanded moment approach. In general, this approach gives better results than the standard approach particularly when the variables (high variance) have low order of power, e.g. two to three, (the order of power could be higher if the variables are of low variance).

Figure 2 shows sample of the Pearson distributions for SP4 and SP6. With lower order power cases, e.g. SP1 to SP3, Pearson distribution of the expanded moment is better than the Pearson obtained from the standard approach when compared to the simulation distribution. However, in higher order powers characterizing a Pearson distribution would be difficult, if not impossible (see SP6 in Fig 2). In nearly all cases, simulation percentile values matched with a Pearson distribution fitted to the 4 moments of the simulation ("Sim M to Pearson").




(a) Expected value (same for both approaches)

(b) Variance

(c) 3rd Moment

(d) 4th moments

Figure 1: Analysis for the sum of variables

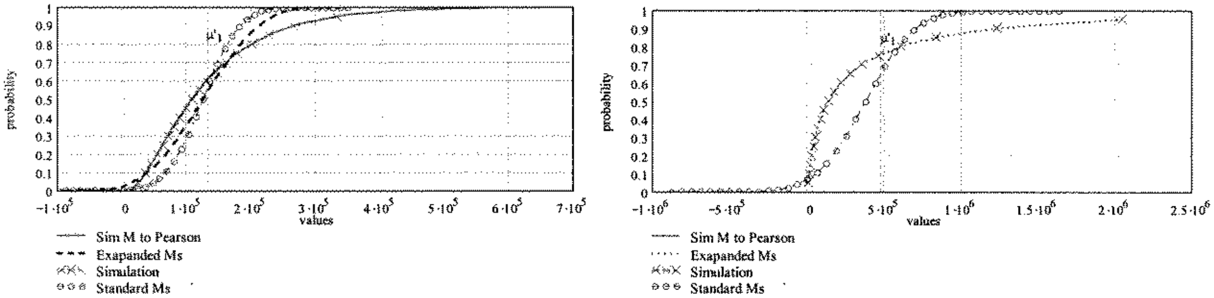



Figure 2: Distributions for the sum of variables in SP4 (left) and SP6 (right)

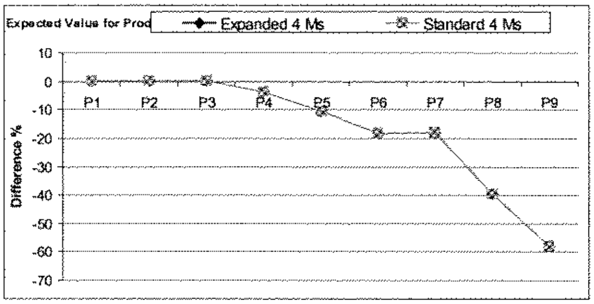
### CASE #2 PRODUCT OF VARIABLES

In addition to the sum of variables, the product of variables is generally used in construction estimates and economics (e.g. uncertain unit rates times resource consumption level). Table 1 shows nine cases for the product of variables where Normal N(10, 5) distribution is used for the variables. The cases are in three groups that differ in the exponential power of the variables. Cases within a group differ in the number of variables. Figure 3 shows the variance, 3$^{rd}$ moments and 4$^{th}$ moments of the standard moments and the expanded four moments as compared to the Monte Carlo simulation for each case. The results show the following. The expected value is reasonably estimated for the first group P1 to P3 where both moment approaches gave exact answers as the simulation; however the accuracy deteriorates for the next two groups from P5 to P9, with P9 deviates by 60% from the simulation result. The increase in the exponential power of the variables and the number of variables in the product increases the deviation of the 2$^{nd}$, 3$^{rd}$, and 4$^{th}$ moments from the simulation results. The expanded moment approach gives better estimates of the variance and skewness moments than the standard moments; however, both deteriorate equally for the third group of higher power and number of variables. The 4$^{th}$ moment in both standard and expanded are not reasonably estimated with the moment approaches for the product of variables.

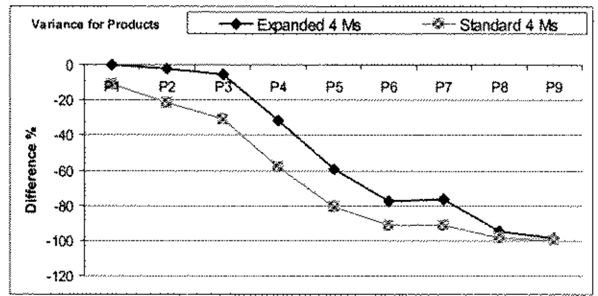
Figure 4 shows sample of the Pearson distributions for P1, P3, P5, and P9. With lower order power cases, e.g. P1, P3, and P5, the Pearson distribution of the expanded moment approaches better the simulation distribution. The increase in the power of variables and their number deviate more the Pearson distribution from the simulation results as seen in P9. It should be noted however when looking at both Figures 3 and 4, and for example P5, that despite the large difference in the moment values between the expanded moment approach and the simulation, the derived Pearson distribution is reasonably close to the simulation results, and that the Pearson of the standard approach is fairly close but with larger tails.

### CASE #3 DIVISION OF VARIABLES

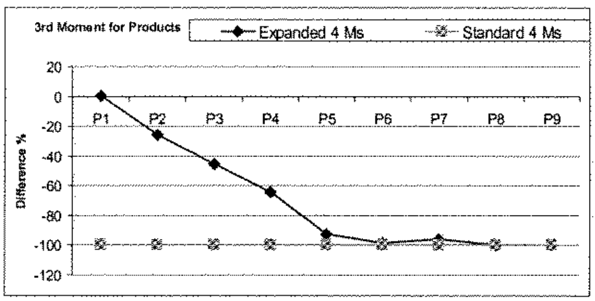
Table 1 shows nine cases similar to those of the product of variables, however with Normal N(100, 10) distribution used for the variables. Similar to previous cases, Figure 5 shows that the increase in the negative power of the variables affects the 2$^{nd}$ to 4$^{th}$ moments; the deviation from the simulation results ranged between 20% and 80%. While the expanded moments give better results, the 4$^{th}$ moment has large deviation compared to the simulation.



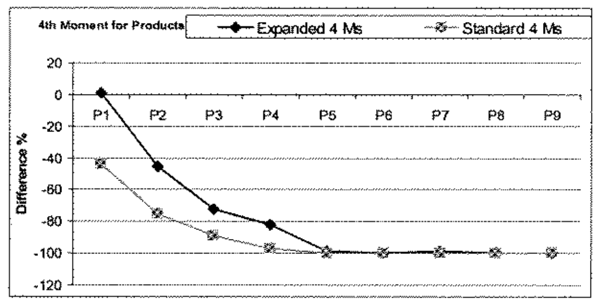

(a) Expected value (same for both approaches)




(b) Variance




(c) 3rd Moment




(d) 4th moments

Figure 3: Analysis for the product of variables

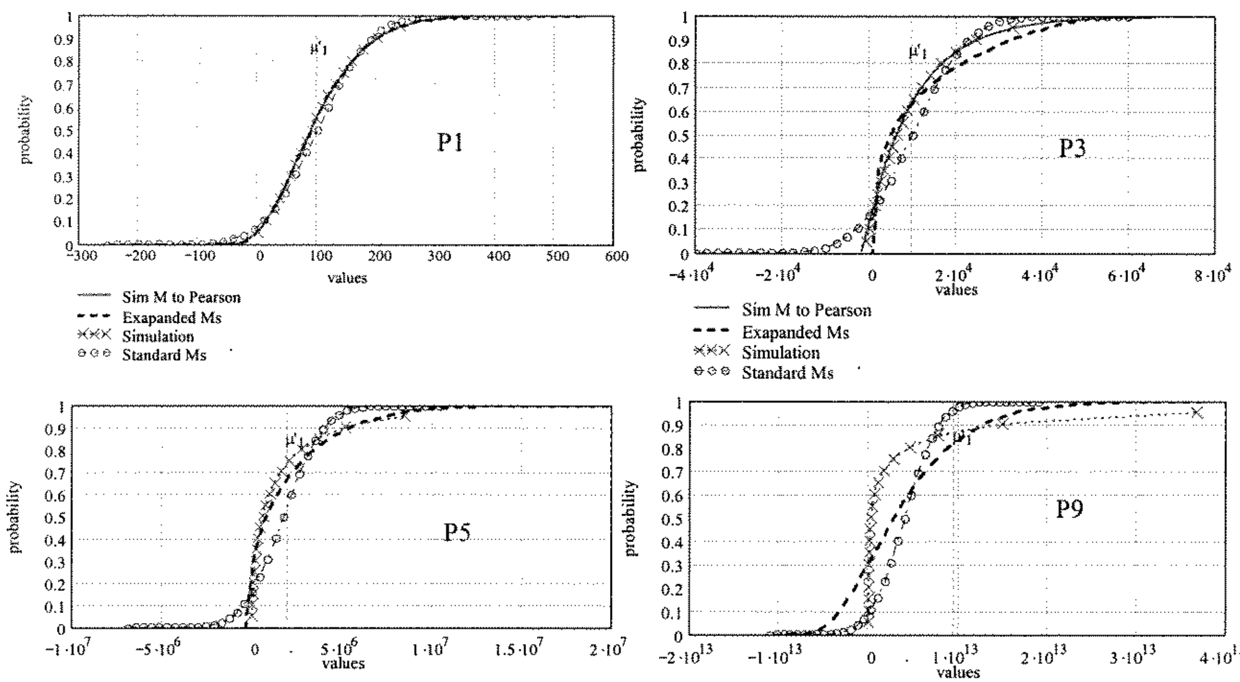
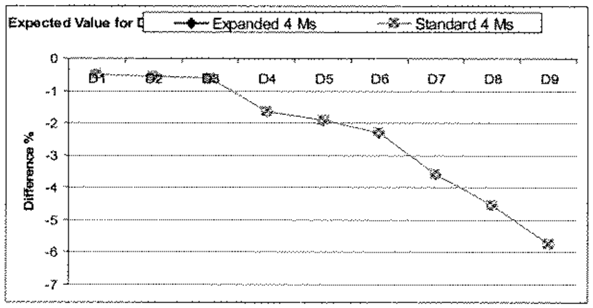
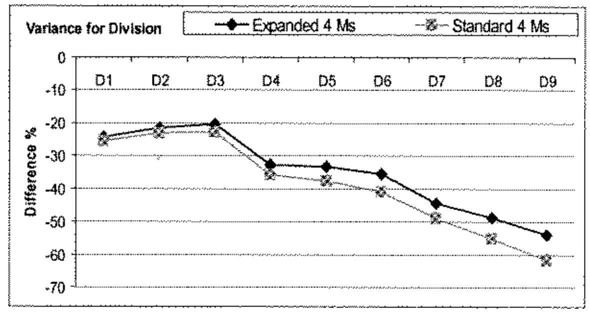


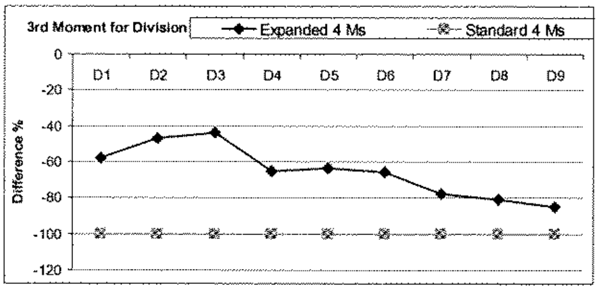

Figure 4: Distributions for the product in P1, P3, P5, and P9



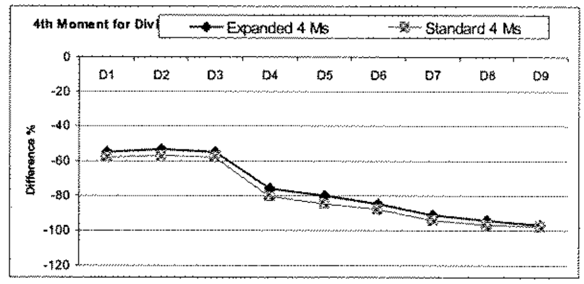

(a) Expected value (same for both approaches)




(b) Variance




(c) 3rd Moment

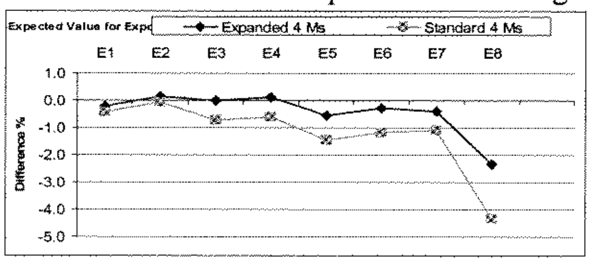



(d) 4th moments

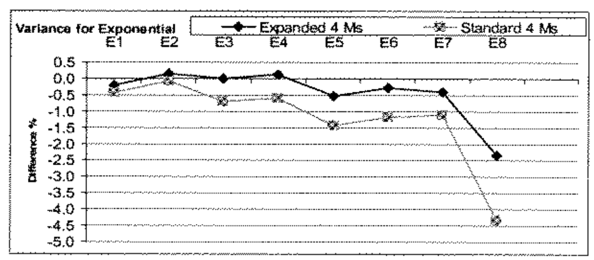
Figure 5: Analysis for the division of variables

### CASE #4 EXPONENTIAL CASES

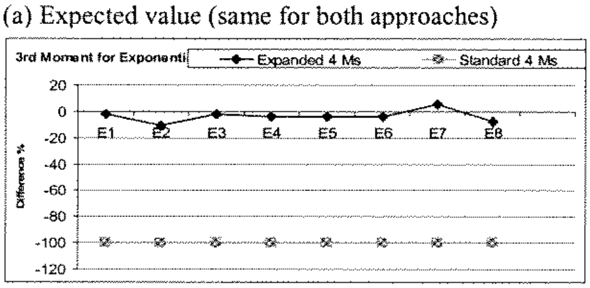
Table 1 shows eight cases with variables multiplied and raised to the natural log e. In these cases, $x_1$ is N(100, 10), $x_2$ is N(0.5, 0.05), $x_3$ is N(0.5, 0.05), and $x_4$ is N(10,5). Figure 6 shows the results of the analysis where the moment approach was a good match to the simulation results. The expanded moments gives better estimates than the standard moments.



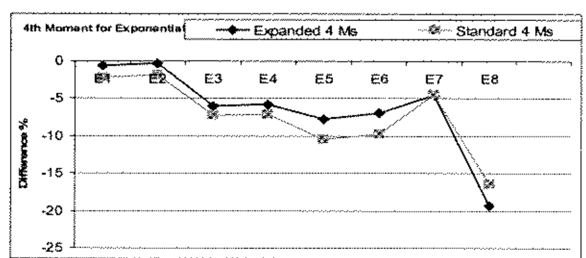

(a) Expected value (same for both approaches)




(b) Variance




(c) 3rd Moment




(d) 4th moments

Figure 6: Analysis for the exponential cases

## CONSTRUCTION EXAMPLE

Here we example a highly abstracted and somewhat artificial representation of a work package related to the construction of the caisson foundations for an elevated rapid transit guideway of some 19.25 km in length in an urban area, with the spacing between foundation and supporting elements being 35 meters, resulting in 551 foundations and pier structures. The example refers specifically to the drilling operation; caissons have diameter of 2 meters and variable depth, depending on soil conditions, between 7 meters to 16 meters. Access to the caissons locations is highly variable, which has a significant impact on set-up and relocation time for the equipment spreads. Due to the rapid price movement and uncertainties associated with geotechnical conditions, the work is done on cost plus basis. Cost to the general contractor is 10% over the subcontractor cost. Basic information is shown in Table 2; the P values are percentiles which were elicited for each variable and followed by fitting processes that produced the probabilistic characteristics shown in the table.

Table 2: Variables of the example

| | Description | P5 | P50 | P95 | Distribut -ion | E[X] | $\sigma$[X] | Skew-ness | kurto sis |
|---|---|---|---|---|---|---|---|---|---|
| N | No. of caissons | 551 | 551 | 551 | - | 551 | | | |
| D | Diameter of caissons, meters | 2 | 2 | 2 | - | 2 | | | |
| d | Depth of caissons, meters | 7 | 10 | 16 | LogN | 10.555 | 2.837 | 0.826 | 4.236 |
| M | Relocation/setup, hours | 6 | 1 | 20 | LogN | 9.85 | 4.545 | 1.483 | 7.148 |
| p | Productivity, $m^3$/crhr | 2 | 8 | 10 | Triangular | 7.333 | 1.929 | -0.454 | 2.4 |
| $C_{or}$ | Unit cost per crew hr, $/crhr | 2500 | 2800 | 3300 | LogN | 2337 | 307.91 | 0.398 | 3.282 |
| R | # of crews | 1 | 1 | 1 | - | 1 | | | |
| P | Profit margin | 0.10 | 0.10 | 0.10 | - | 0.1 | | | |
| $C_{oi}$ | Indirect, $/yr | 275,000 | 350,000 | 450,000 | Beta | 354639 | 54089 | 0.2997 | 2.194 |

The time $T_i$ required for the i[th] caisson and total time $T$ and Cost $C_o$ for all caissons are

$$T_i = M_i + \frac{Q_i}{p_i \cdot R} \text{ in which } Q_i = \pi \cdot \left(\frac{D}{2}\right)^2 \cdot d_i \tag{5}$$

$$T = \frac{1}{1920} \cdot \sum M_i + \frac{Q_i}{p_i \cdot R} = \frac{N}{1920} \cdot \left( M + \frac{Q}{p \cdot R} \right) \text{ (yrs, 8 hrs/day, 5 days/week, 48 wks/year)} \tag{6}$$

$$C_o = (1+P) \cdot \left[ N \cdot C_{or} \cdot \left( M + \frac{Q}{p \cdot R} \right) + C_{oi} \cdot T \right] \tag{7}$$

In the example the variables M, Q, and p are assumed to be independent and identically distributed for all i (551 locations). One crew is used through out the drilling operations.

The above time and cost formulations reflect some of the simple mathematical operations described before. Performing the analysis on the time and cost of all caissons using the standard and expanded moment analysis as well as Monte Carlo simulation of 100,000 iterations shows that the expanded moments have better results than the standard moments when compared to simulation, Figure 7. Figure 8 shows how the Pearson distributions for the time and cost coincided with that obtained using Monte Carlo simulation.

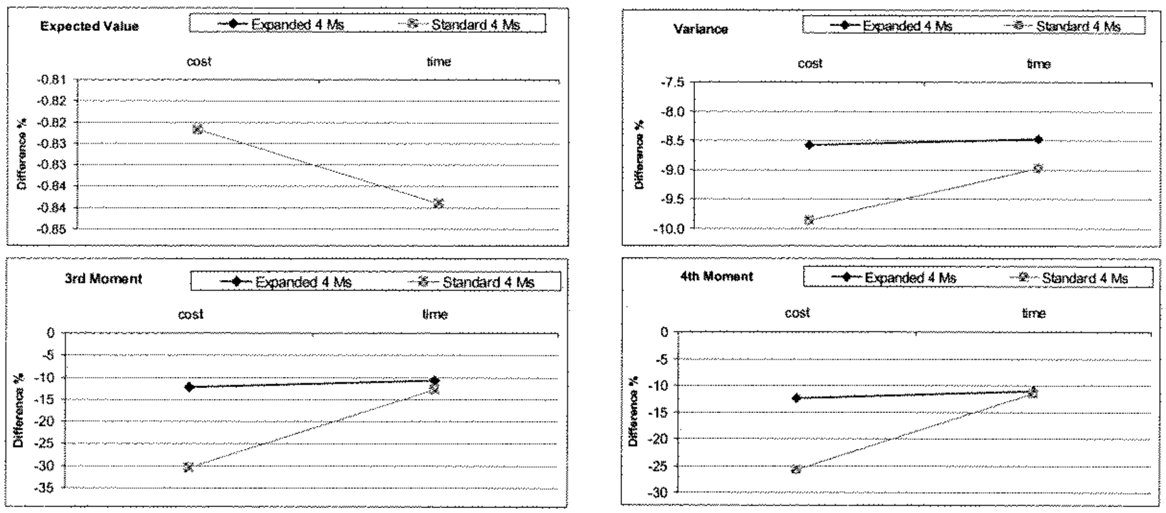



Figure 7: Analysis of the time and cost for caisson operations

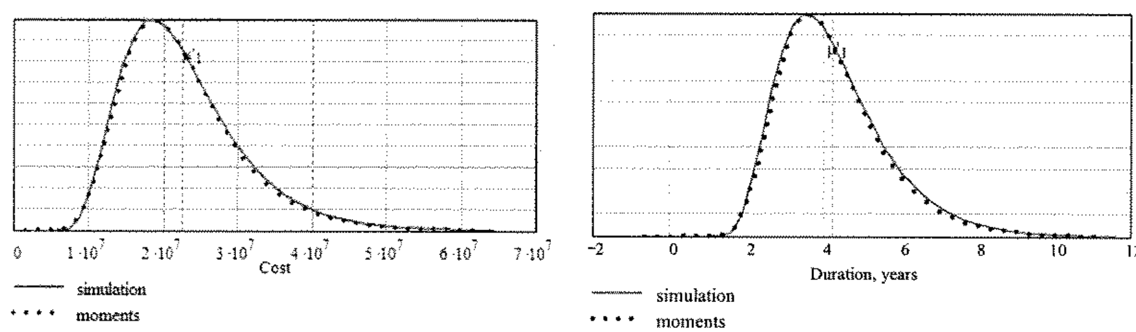



Figure 8: Pearson and simulation distributions for the time and cost

## SUMMARY AND CONCLUSIONS

The above analysis explained the performance of both of a standard moment approach and a new expanded moment approach under several cases of sum, multiplication, division and exponentiation. Four moments of each case and their characterized Pearson distribution were compared to their equivalents of a large scale Monte Carlo simulation.

The results of the analysis shows that both of the moment approaches could provide reasonable estimates for construction and economic estimates conditioned on that the formulation of these estimates is simple where the power/exponent of the variables is not large. If the formulation encountered higher order of power the moment approach would be used with cautions.

The analysis shows that the use of the expanded derivation of the four moments produces better results than the standard four moments, particularly for the third moment/skewness of performance measures. However, when experiencing variables with higher order of power, both moment approaches largely deviates from near exact results of Monte Carlo simulation.

The use of the Pearson distribution family is significant to the moment approach where closed form distributions and probabilistic estimates could be obtained. For construction estimating and economics purposes, the use of Pearson would be fairly reasonable particularly if the probabilities at the distribution tail are not of major interests to construction decision makers. That is unlike risk analysis for safety and failure analysis where distribution tail is of major importance. When working with cases that have variables of higher powers, the use of moments might be questioned. For these cases, while the four moments could be obtained, characterizing a distribution from the Pearson family would be difficult, if not impossible. In such cases, another distribution family would be consulted or the analysis could be conducted using Monte Carlo simulation.

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**Appendix A**

Taylor series expansion of a second order about the mean values of $\boldsymbol{x}$ as follows

$$f(\mathbf{x}) = f(\overline{\mathbf{x}}) + \sum_{i=1}^{n} f_i' \cdot \delta_i + \frac{1}{2} \cdot \sum_{i=1}^{n} f_i'' \cdot \delta_i^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_{i,j}' \cdot \delta_i \cdot \delta_j \tag{A.1}$$

Expected value:

$$\begin{aligned} \mu_1'(Z) &= E[f(\mathbf{x})] \\ &= f(\overline{\mathbf{x}}) + \frac{1}{2} \cdot \sum_{i=1}^{n} f_i'' \cdot \mu 2_i \end{aligned} \tag{A.2}$$

Second central moment:

$$\begin{aligned} \mu_2(Z) &= E[(Z - \mu_1'(Z))^2] \\ &= E[(f(\mathbf{x}) - E[f(\mathbf{x})])^2] \\ &= E\left\{\left(\sum_{i=1}^{n} f_i' \cdot \delta_i + \frac{1}{2} \cdot \sum_{i=1}^{n} f_i'' \cdot \delta_i^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_{i,j}' \cdot \delta_i \cdot \delta_j - \frac{1}{2} \cdot \sum_{i=1}^{n} f_i'' \cdot \mu 2_i\right)^2\right\} \\ &= E\left\{(H + I + J + K)^2\right\} \end{aligned} \tag{A.3}$$

$$\begin{aligned} \mu_2(Z) &= E[H^2] + E[I^2] + E[J^2] + E[K^2] \\ &\quad + E[2HI] + E[2HJ] + E[2HK] + E[2IJ] + E[2HK] + E[2JK] \end{aligned} \tag{A.4}$$

where,

$$E[H^2] = \sum_{i=1}^{n} f_i'^2 \cdot \mu 2_i \tag{A.5}$$

$$E[I^2] = \frac{1}{4} \cdot \sum_{i=1}^{n} f_i''^2 \cdot \mu 4_i + \frac{1}{2} \cdot \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_i'' \cdot f_j'' \cdot \mu 2_i \cdot \mu 2_j \tag{A.6}$$

$$E[J^2] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_{i,j}'^2 \cdot \mu 2_i \cdot \mu 2_j \tag{A.7}$$

$$E[K^2 + 2IK] = -\frac{1}{4} \cdot \left(\sum_{i=1}^{n} f_i'' \cdot \mu 2_i\right)^2 \tag{A.8}$$

$$E[2HI] = \sum_{i=1}^{n} f_i' \cdot f_j'' \cdot \mu 3_i \tag{A.9}$$

Third central moment:

$$\begin{aligned} \mu_3(Z) &= E[(Z - \mu_1'(Z))^3] \\ &= E[(f(\mathbf{x}) - E[f(\mathbf{x})])^3] \\ &= E\left\{\left(\sum_{i=1}^{n} f_i' \cdot \delta_i + \frac{1}{2} \cdot \sum_{i=1}^{n} f_i'' \cdot \delta_i^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_{i,j}' \cdot \delta_i \cdot \delta_j - \frac{1}{2} \cdot \sum_{i=1}^{n} f_i'' \cdot \mu 2_i\right)^3\right\} \\ &= E\left\{(H + I + J + K)^3\right\} \end{aligned} \tag{A.10}$$

$$
\begin{aligned}
\mu_3(Z) = {} & E[H^3] + E[I^3] + E[J^3] + E[K^3] \\
& + E[3H^2 I] + E[3H^2 J] + E[3H^2 K] + E[3HI^2] + E[3HJ^2] + E[3HK^2] + E[6HIJ] + E[6HIK] \\
& + E[3I^2 J] + E[3I^2 K] + E[3IJ^2] + E[3IK^2] + E[6IJk\ ] \\
& + E[3J^2 K] + E[3JK^2]
\end{aligned}
\tag{A.11}
$$

$$
E[H^3] = \sum_{i=1}^{n} f_i'^3 \cdot \mu 3_i \tag{A.12}
$$

$$
\begin{aligned}
E[I^3] = {} & \frac{3}{8} \cdot \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_i''^2 \cdot f_j'' \cdot \mu 4_i \cdot \mu 2_j + \frac{3}{8} \cdot \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_i'' \cdot f_j''^2 \cdot \mu 2_i \cdot \mu 4_j \\
& + \frac{6}{8} \cdot \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} f_i'' \cdot f_j'' \cdot f_k'' \cdot \mu 2_i \cdot \mu 2_j \cdot \mu 2_k
\end{aligned}
\tag{A.13}
$$

$$
E[J^3] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_{i,j}'^3 \cdot \mu 3_i \cdot \mu 3_j \tag{A.14}
$$

$$
E[K^3 + 3IK^2] = \frac{1}{4} \cdot \left( \sum_{i=1}^{n} f_i'' \cdot \mu 2_i \right)^3 \tag{A.15}
$$

$$
E[3H^2 I] = \frac{3}{2} \cdot \left[ \sum_{i=1}^{n} f_i'^2 \cdot f_i'' \cdot \mu 4_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_i'^2 \cdot f_i'' \cdot \mu 2_i \cdot \mu 2_j + \sum_{i=2}^{n} \sum_{j=1}^{i-1} f_i'^2 \cdot f_i'' \cdot \mu 2_i \cdot \mu 2_j \right] \tag{A.16}
$$

$$
E\left[ 6H^2 J \right] = 6 \cdot \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_i' \cdot f_j' \cdot f_{i,j}' \cdot \mu 2_i \cdot \mu 2_j \tag{A.17}
$$

$$
E\left[ 3H^2 K \right] = \left( \sum_{i=1}^{n} f_i'' \cdot \mu 2_i \right) \cdot \left[ \frac{-3}{2} \cdot \sum_{i=1}^{n} f_i'^2 \cdot \mu 2_i \right] \tag{A.18}
$$

$$
E[3H\, I^2] = \frac{3}{4} \cdot \left[ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_i' \cdot f_i'' \cdot f_j'' \cdot \mu 3_i \cdot \mu 2_j + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_j' \cdot \cdot f_i'' \cdot f_j'' \cdot \mu 2_i \cdot \mu 3_j \right] \tag{A.19}
$$

$$
E[3H\, J^2] = 3 \cdot \left[ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_i' \cdot f_{i,j}'^2 \cdot \mu 3_i \cdot \mu 2_j + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_j' \cdot f_{i,j}'^2 \cdot \mu 2_i \cdot \mu 3_j \right] \tag{A.20}
$$

$$
E\left[ 3I^2 J \right] = \frac{3}{2} \cdot \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_i'' \cdot f_j'' \cdot f_{i,j}' \cdot \mu 3_i \cdot \mu 3_j \tag{A.21}
$$

$$
E[6HIK] = \left( \sum_{i=1}^{n} f_i'' \cdot \mu 2_i \right) \cdot \left[ \frac{-3}{2} \cdot \sum_{i=1}^{n} f_i' \cdot f_i'' \cdot \mu 3_i \right] \tag{A.22}
$$

$$
E[3I^2 K] = \left( \sum_{i=1}^{n} f_i'' \cdot \mu 2_i \right) \cdot \left[ \frac{-3}{8} \cdot \sum_{i=1}^{n} f_i''^2 \cdot \mu 4_i + \frac{-3}{4} \cdot \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_i'' \cdot f_j'' \cdot f_{i,j}' \cdot \mu 2_i \cdot \mu 2_j \right] \tag{A.23}
$$

$$
E[3J^2 K] = \left( \sum_{i=1}^{n} f_i'' \cdot \mu 2_i \right) \cdot \left[ \frac{-3}{2} \cdot \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_{i,j}'^2 \cdot \mu 2_i \cdot \mu 2_j \right] \tag{A.24}
$$

$$
E[3I\, J^2] = \frac{3}{2} \cdot \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_i'' \cdot f_{i,j}'^2 \cdot \mu 4_i \cdot \mu 2_j + \frac{3}{2} \cdot \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_j'' \cdot f_{i,j}'^2 \cdot \mu 2_i \cdot \mu 4_j \tag{A.25}
$$

Fourth central moment:

$$\mu_4(Z) = E[(Z-\mu_1'(Z))^4] = E[(f(\mathbf{x})-E[f(\mathbf{x})])^4]$$

$$= E\left\{\left(\sum_{i=1}^{n} f_i'\cdot\delta_i + \frac{1}{2}\cdot\sum_{i=1}^{n} f_i''\cdot\delta_i^2 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_{i,j}'\cdot\delta_i\cdot\delta_j - \frac{1}{2}\cdot\sum_{i=1}^{n} f_i''\cdot\mu 2_i\right)^4\right\} = E\left\{(H+I+J+K)^4\right\} \tag{A.26}$$

$$\begin{aligned}\mu_4(Z) = {} & E[H^4]+E[I^4]+E[J^4]+E[K^4] \\ & + E[4H^3I]+E[4H^3J]+E[4H^3K]+E[6H^2I^2]+E[6H^2J^2]+E[6H^2K^2] \\ & + E[12H^2IJ]+E[12H^2IK]+E[12H^2JK]+E[12HI^2J]+E[12HI^2K]+E[12HJ^2K] \\ & + E[12HIJ^2]+E[12HIK^2]+E[12HJK^2]+E[24HIJK]+E[4HI^3]+E[4HJ^3]+E[4HK^3] \\ & + E[4I^3J]+E[4I^3K]+E[6I^2J^2]+E[6I^2K^2]+E[12I^2JK]+E[12IJ^2K]+E[12IJK^2] \\ & + E[4IJ^3]+E[4IK^3]+E[4J^3K]+E[6J^2K^2]+E[4JK^3]\end{aligned} \tag{A.27}$$

$$E[H^4] = \sum_{i=1}^{n} f_i'^4\cdot\mu 4_i + 6\cdot\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i'^2\cdot f_j'^2\cdot\mu 2_i\cdot\mu 2_j \tag{A.28}$$

$$\begin{aligned}E[I^4] = {} & \frac{6}{16}\cdot\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i''^2\cdot f_j''^2\cdot\mu 4_i\cdot\mu 4_j + \frac{12}{16}\cdot\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} f_i''^2\cdot f_j''\cdot f_k''\cdot\mu 4_i\cdot\mu 2_j\cdot\mu 2_k \\ & + \frac{12}{16}\cdot\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} f_i''\cdot f_j''^2\cdot f_k''\cdot\mu 2_i\cdot\mu 4_j\cdot\mu 2_k + \frac{12}{16}\cdot\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} f_i''\cdot f_j''\cdot f_k''^2\cdot\mu 2_i\cdot\mu 2_j\cdot\mu 4_k\end{aligned} \tag{A.29}$$

$$\begin{aligned}E[J^4] = {} & \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_{i,j}'^4\cdot\mu 4_i\cdot\mu 4_j + 6\cdot\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} f_{i,j}'^2\cdot f_{j,k}'^2\cdot\mu 2_i\cdot\mu 4_j\cdot\mu 2_k + 6\cdot\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} f_{i,k}'^2\cdot f_{j,k}'^2\cdot\mu 2_i\cdot\mu 2_j\cdot\mu 4_k \\ & + 6\cdot\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} f_{i,j}'^2\cdot f_{j,k}'^2\cdot\mu 4_i\cdot\mu 2_j\cdot\mu 2_k + 12\cdot\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} f_{i,j}'\cdot f_{i,k}'^2\cdot f_{j,k}'\cdot\mu 3_i\cdot\mu 2_j\cdot\mu 3_k \\ & + 12\cdot\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} f_{i,j}'^2\cdot f_{i,k}'\cdot f_{j,k}'\cdot\mu 3_i\cdot\mu 3_j\cdot\mu 2_k + 12\cdot\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} f_{i,j}'\cdot f_{i,k}'\cdot f_{j,k}'^2\cdot\mu 2_i\cdot\mu 3_j\cdot\mu 3_k\end{aligned} \tag{A.30}$$

$$E[K^4+4IK^3] = \frac{-3}{16}\cdot\left(\sum_{i=1}^{n} f_i''\cdot\mu 2_i\right)^4 \tag{A.31}$$

$$E[4H^3I] = 2\cdot\left[\sum_{i=1}^{n}\sum_{j=1}^{n} f_i'^3\cdot f_j''\cdot\mu 3_i\cdot\mu 2_j\right] \tag{A.32}$$

$$E\left[4H^3J\right] = 12\cdot\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i'^2\cdot f_j'\cdot f_{i,j}'\cdot\mu 3_i\cdot\mu 2_j + 12\cdot\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i'\cdot f_j'^2\cdot f_{i,j}'\cdot\mu 2_i\cdot\mu 3_j \tag{A.33}$$

$$E\left[4H^3K\right] = \left[\sum_{i=1}^{n} f_i''\cdot\mu 2_i\right]\cdot\left[-2\cdot\sum_{i=1}^{n} f_i'^3\cdot\mu 3_i\right] \tag{A.34}$$

$$E\left[6H^2I^2\right] = \frac{6}{4}\cdot\left[\sum_{i=1}^{n}\sum_{j=1}^{n} f_i'^2\cdot f_j''^2\cdot\mu 2_i\cdot\mu 4_j + 4\cdot\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i'\cdot f_j'\cdot f_i''\cdot f_j''\cdot\mu 3_i\cdot\mu 3_j\right] \tag{A.35}$$

$$E\left[4H^2J^2\right] = 12\cdot\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i'\cdot f_j'\cdot f_{i,j}'^2\cdot\mu 3_i\cdot\mu 3_j + 6\cdot\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i'^2\cdot f_{i,j}'^2\cdot\mu 4_i\cdot\mu 2_j + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_j'^2\cdot f_{i,j}'^2\cdot\mu 2_i\cdot\mu 4_j\right] \tag{A.36}$$

$$E\left[6H^2K^2\right] = \left[\sum_{i=1}^{n} f_i''\cdot\mu 2_i\right]^2\cdot\left[\frac{3}{2}\cdot\sum_{i=1}^{n} f_i'^2\cdot\mu 2_i\right] \tag{A.37}$$

$$E\left[12H^2IJ\right] = 12\cdot\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i''\cdot f_i'\cdot f_{i,j}'\cdot\mu 4_i\cdot\mu 2_j + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_j''\cdot f_i'\cdot f_{i,j}'\cdot\mu 2_i\cdot\mu 4_j + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i'^2\cdot f_j''\cdot f_{i,j}'\cdot\mu 3_i\cdot\mu 3_j\right] \tag{A.38}$$

$$E\left[12H^2IK\right] = \left(-3\cdot\sum_{i=1}^{n} f_i''\cdot\mu 2_i\right)\cdot\left[\sum_{i=1}^{n} f_i'^2\cdot f_i''\cdot\mu 4_i + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i'^2\cdot f_i''\cdot\mu 2_i\cdot\mu 2_j + \sum_{i=2}^{n}\sum_{j=1}^{i-1} f_i'^2\cdot f_i''\cdot\mu 2_i\cdot\mu 2_j\right] \tag{A.39}$$

$$E\left[12H^2JK\right] = \left(-12\cdot\sum_{i=1}^{n} f_i''\cdot\mu 2_i\right)\cdot\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i'\cdot f_j'\cdot f_{i,j}'\cdot\mu 2_i\cdot\mu 2_j\right] \tag{A.40}$$

$$E\left[12HI^2J\right] = 6\cdot\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i'\cdot f_i'' f_j''\cdot f_{i,j}'\cdot\mu 4_i\cdot\mu 3_j + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_j'\cdot f_i'' f_j''\cdot f_{i,j}'\cdot\mu 3_i\cdot\mu 4_j\right] \tag{A.41}$$

$$E\left[12HI^2K\right] = -3\cdot\left(\sum_{i=1}^{n} f_i''\cdot\mu 2_i\right)\cdot\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i'\cdot f_i'' f_j''\cdot\mu 3_i\cdot\mu 2_j + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_j'\cdot f_i'' f_j''\cdot\mu 2_i\cdot\mu 3_j\right] \tag{A.42}$$

$$E\left[12HJ^2K\right] = -6\cdot\left(\sum_{i=1}^{n} f_i''\cdot\mu 2_i\right)\cdot\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i'\cdot f_{i,j}'^2\cdot\mu 3_i\cdot\mu 2_j + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_j'\cdot f_{i,j}'^2\cdot\mu 2_i\cdot\mu 3_j\right] \tag{A.43}$$

$$E\left[12HIJ^2\right] = \frac{12}{2}\cdot\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i'\cdot f_j''\cdot f_{i,j}'^2\cdot\mu 3_i\cdot\mu 4_j + \sum_{i=1}^{n}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} f_i'\cdot f_i''\cdot\mu 3_i\cdot f_{j,k}'^2\cdot\mu 2_j\cdot\mu 2_k\right] \tag{A.44}$$

$$E\left[12HIK^2\right]=\left(\sum_{i=1}^{n} f_i''\cdot \mu 2_i\right)^2\cdot\left[\frac{3}{2}\cdot\sum_{i=1}^{n-1} f_i'\cdot f_i''\cdot \mu 3_i\right] \tag{A.45}$$

$$E[24HIJK]=\frac{-24}{4}\cdot\left(\sum_{i=1}^{n} f_i''\cdot \mu 2_i\right)\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i'\cdot f_j''\cdot f_{i,j}'\cdot \mu 2_i\cdot \mu 3_j\right] \tag{A.46}$$

$$E\left[4HI^3\right]=\frac{12}{8}\cdot\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_j'\cdot f_i''^2\cdot f_j''\cdot \mu 4_i\cdot \mu 3_j+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i'\cdot f_i''^2\cdot f_j''\cdot \mu 3_i\cdot \mu 4_j\right] \tag{A.47}$$

$$E\left[4HJ^3\right]=4\cdot\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i'\cdot f_{i,j}'^3\cdot \mu 4_i\cdot \mu 3_j+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_j'\cdot f_{i,j}'^3\cdot \mu 3_i\cdot \mu 4_j\right] \tag{A.48}$$

$$E\left[4I^3K\right]=\left(\frac{-1}{4}\cdot\sum_{i=1}^{n} f_i''\cdot \mu 2_i\right)\cdot\left[3\cdot\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i''^2\cdot f_j''\cdot \mu 4_i\cdot \mu 2_j+3\cdot\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i''\cdot f_j''^2\cdot \mu 2_i\cdot \mu 4_j+\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} f_i''\cdot f_j''\cdot f_k''\cdot \mu 2_i\cdot \mu 2_j\cdot \mu 2_k\right] \tag{A.49}$$

$$E\left[6I^2J^2\right]=\frac{6}{2}\cdot\sum_{i=1}^{n}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} f_i''\cdot f_{j,k}'^2\cdot \mu 2_i\cdot \mu 2_j\cdot \mu 2_k+6\cdot\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i''\cdot f_j''\cdot f_{i,j}'^2\cdot \mu 4_i\cdot \mu 4_j \tag{A.50}$$

$$E\left[6I^2K^2\right]=\left(\sum_{i=1}^{n} f_i''\cdot \mu 2_i\right)^2\cdot\left[\frac{3}{8}\cdot\sum_{i=1}^{n-1} f_i''^2\cdot \mu 4_i+\frac{3}{4}\cdot\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i''\cdot f_j''\cdot \mu 2_i\cdot \mu 2_j\right] \tag{A.51}$$

$$E\left[12I^2JK\right]=\left(-\sum_{i=1}^{n} f_i''\cdot \mu 2_i\right)\left[3\cdot\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i''\cdot f_j''\cdot f_{i,j}'\cdot \mu 3_i\cdot \mu 3_j\right] \tag{A.52}$$

$$E\left[12IJ^2K\right]=\left(-3\cdot\sum_{i=1}^{n} f_i''\cdot \mu 2_i\right)\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_i''\cdot f_{i,j}'^2\cdot \mu 4_i\cdot \mu 2_j+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_j''\cdot f_{i,j}'^2\cdot \mu 2_i\cdot \mu 4_j\right] \tag{A.53}$$

$$E\left[4IJ^3\right]=2\cdot\sum_{i=1}^{n}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} f_i''\cdot f_{j,k}'^3\cdot \mu 2_i\cdot \mu 3_j\cdot \mu 3_k+12\cdot\sum_{i=1}^{n}\sum_{j=1}^{n-2}\sum_{k=j+1}^{n-1}\sum_{m=k+1}^{n} f_i''\cdot \mu 2_i\cdot f_{j,k}'\cdot \mu 2_j\cdot \mu 2_k\cdot f_{j,m}'\cdot \mu 2_m\cdot f_{k,m}' \tag{A.54}$$

$$E\left[4J^3K\right]=\left(-\sum_{i=1}^{n} f_i''\cdot \mu 2_i\right)\cdot\left[2\cdot\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_{i,j}'^3\cdot \mu 3_i\cdot \mu 3_j\right] \tag{A.55}$$

$$E\left[6J^2K^2\right]=\left(-\sum_{i=1}^{n} f_i''\cdot \mu 2_i\right)^2\cdot\left[\frac{3}{2}\cdot\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_{i,j}'^2\cdot \mu 2_i\cdot \mu 2_j\right] \tag{A.56}$$